Economics 1011a
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Section 0: Math Review

1. Partial Derivatives
2. Total Differential
3. Implicit Functions
4. Constrained Maximization: Lagrangian and Lagrange Multipliers

1. Partial Derivatives

Definition: Given a function of two variables $f(x, y)$, we define its partial derivative with respect to (say) x at the point $(\bar{x}, \bar{y})$ the quantity:

$$\frac{\partial f}{\partial x}(\bar{x}, \bar{y}) = \lim_{h \to 0} \frac{f(\bar{x}+h, \bar{y}) - f(\bar{x}, \bar{y})}{h} \tag{1}$$

Remarks: The partial derivative of $f(x, y)$ with respect x:

- is defined to be the derivative of the function $f(x, \bar{y})$ for *fixed* $\bar{y}$ (so that $f(x, \bar{y})$ is a function of x only).
- can be intuitively described as the marginal increment of f *along* the x-axis (use a picture)
- is still a function of (x, y).

Example: If:

$$f(x, y) = 3x^3 + 2x^2y + 4y^2$$

Then

$$\frac{\partial f}{\partial x} = 9x^2 + 4xy; \qquad \frac{\partial f}{\partial y} = 2x^2 + 8y$$

Notice that the partial derivative is still a function, we can therefore take the partial derivative of a partial derivative (which is called *second order* partial derivative) as follows (in the example):

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial[9x^2 + 4xy]}{\partial x} = 18x + 4y; \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial[2x^2 + 8y]}{\partial y} = 8$$

and

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial[2x^2 + 8y]}{\partial x} = 4x = \frac{\partial[9x^2 + 4xy]}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

Notice that the second order cross partial derivatives are the same. This is indeed always the case (under very general conditions). The theorem which proves this fact is called **Young's Theorem**.

2. Total Differential

Given a function of n variables $f(x_1, .., x_n)$, we define the *total differential* by:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + ... + \frac{\partial f}{\partial x_n} dx_n = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

So, for example, if we have a function:

$$f(x, y, z) = x^\alpha y^\beta + \sqrt{z}$$

we have that its total differential is:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \tag{2}$$

$$= \alpha x^{\alpha - 1} y^\beta dx + \beta x^\alpha y^{\beta - 1} dy + \frac{1}{2\sqrt{z}} dz \tag{3}$$

3. Implicit Functions

Notice that contour lines (indifference curves in utility theory) of a function $f(x, y)$ have the following equation:

$$f(x_1, x_2) = k \tag{4}$$

where k is a constant (the "utility level"). Also, this last expression *implicitly* defines a function

$$x_2 = g(x_1)$$

that is, a function that assigns a value of x_2 at every x_1. To find the derivative

$$\frac{dx_1}{dx_2} = g'(x_1)$$

we can simply take the total differential of (4). Notice that on the RHS we have zero, while on the LHS we have $\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$, so that:

$$\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

which implies

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}$$

In utility theory we call $-\frac{dx_2}{dx_1}$ *Marginal Rate of Substitution*

4. Un/Constrained Maximization

Recall that the FOC (*Necessary* Condition) for a *un*constrained maximum problem:

$$\max_x f(x)$$

is

$$f'(x) = 0.$$

The sufficient condition for a *maximum* is

$$f''(x) < 0.$$

For a *minimum* is instead

$$f'' > 0.$$

(Give quick intuition since we have $df = f'(x)dx$, if x is optimum, we *must* have $df \leq 0$. Since dx can be positive or negative, it must be the case that at an optimum $f'(x) = 0$.)

In the multivariable case, we have:

$$\max_{x_1,...,n_n} f(x_1, ..., x_n)$$

and FOC are

$$\frac{\partial f}{\partial x_i} = 0 \quad \forall i = 1, .., n.$$

(Again, intuition is that we can write $df = \sum f_{x_i} dx_i$, if a vector $(x_1, .., x_n)$ is a maximum, we must have $df \leq 0$. Since we can choose the dx_is, it must be the case that $f_{x_i} = 0$ for all i)

The sufficient conditions for a *maximum* or a *minimum* are analogous to the ones for the one-variable case, but they are a bit more involved and are not given here.

For *constrained maximization* problems:

$$\max_{x_1,..,x_n} f(x_1, .., x_n)$$

subject to

$$g(x_1, ..., x_n) = c \tag{5}$$

we form the **Lagrangian**:

$$\mathcal{L}(x_1, .., x_n, \lambda) = f(x_1, .., x_n) + \lambda(c - g(x_1, ..., x_n))$$

and then we just maximize it, that is we find its FOC:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = c - g(x_1, ..., x_n) = 0$$

Again, second order conditions exist but we do not require their explicit knowledge here.

Intuition for the Lagrangian Method: suppose you have the following max problem:

$$\max_{x,y} f(x, y) \quad \text{s.t.} \quad g(x, y) = c$$

As we have seen, $g(x, y) = c$ defines (under general conditions) a function $y = G(x, c)$ and we know that $dy/dx = G_x = -g_x/g_y$. So, substitute for y into f to get

$$\max_{x} f(x, G(x, c))$$

and take FOC:

$$f_x + f_y G_x = 0 \qquad \Longleftrightarrow \qquad f_x - f_y \frac{g_x}{g_y} = 0$$

Define

$$\lambda = \frac{f_y}{g_y}$$

to get

$$f_x - \lambda g_x = 0$$

Since we could do all the above considering a function $x = H(y, c)$, we have that both the FOC we have seen must hold.

<u>On the interpretation of the Lagrange multiplier λ:</u>

It can be interpreted as the marginal gain in the function f for a marginal change in the constraint c. Consider for example the following trivial example with one variable and one constraint:

$$\max_x u(x) = \sqrt{x}$$

$$\text{s.t. } x = I = 36$$

The solution to this max problem is obvious, but let's solve it by using the method of Lagrange Multipliers:

$$\mathcal{L} = \sqrt{x} + \lambda(36 - x)$$

so that

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2\sqrt{x}} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 36 - x = 0$$

so that the solution is (obviously) $x = 36$. From the first equation we also get:

$$\lambda = \frac{1}{2\sqrt{36}} = \frac{1}{2 \cdot 6} = \frac{1}{12} \sim 0.083$$

What if $I = 37$? We would get the solution is $x = 37$ and

$$u(37) = \sqrt{37} \sim 6.083 = u(36) + \lambda$$

Notice then that, as we said, λ gives the marginal increment in the *optimal* value of the objective function for a marginal increment in the constraint.

In general, notice that from the FOC to the problem we have

$$\frac{f_{x_1}}{g_{x_1}} = \frac{f_{x_2}}{g_{x_2}} = ... = \frac{f_{x_n}}{g_{x_n}} = \lambda$$

To interpret this equation, consider a marginal increment in c, say dc. Taking the total differential of the constraint we get:

$$\sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i = dc$$

and suppose now that we decide to move only the variable x_1. So we have:

$$\begin{aligned} \partial n_i &= 0 \text{ for } i \geq 2 \\ \frac{\partial g}{\partial x_1} dx_1 &= dc \end{aligned}$$

or

$$\frac{dx_1}{dc} = \frac{1}{g_{x_1}}$$

Hence we get:

$$\lambda = \frac{f_{x_1}}{g_{x_1}} = f_{x_1} \frac{dx_1}{dc}$$

which is the marginal benefit to f via a change in x_1 due to a marginal change in c. Hence, at an optimum we must have that the for a given marginal change in c the marginal benefit to f is the same for every variable x_i and, in particular, equal to λ.