ECON 117 MATHEMATICAL HANDOUT

Winter 2007 Mark Machina

A. CALCULUS REVIEW[1]

You should already know what a derivative is. We'll use the expressions $f'(x)$ or $df(x)/dx$ for the derivative of the function $f(x)$. To indicate the derivative of $f(x)$ evaluated at the point $x = x^*$, we'll use the expressions $f'(x^*)$ or $df(x^*)/dx$.

When we have a function of more than one variable, we can consider its derivatives with respect to each of the variables, that is, each of its **partial derivatives**. We use the expressions

$$\frac{\partial f(x_1,x_2)}{\partial x_1} \quad \text{and} \quad f_1(x_1,x_2)$$

interchangeably to denote the partial derivative of $f(x_1,x_2)$ with respect to its first argument (that is, with respect to x_1). To calculate this, just hold x_2 fixed (treat it as a constant) so that $f(x_1,x_2)$ may be thought of as a function of x_1 alone, and differentiate it with respect to x_1. The notation for partial derivatives with respect to x_2 (or in the general case, with respect to x_i) is analogous.

For example, if $f(x_1,x_2) = x_1^2 x_2 + 3x_1$, we have

$$\frac{\partial f(x_1,x_2)}{\partial x_1} = f_1(x_1,x_2) = 2x_1 \cdot x_2 + 3 \qquad \text{and} \qquad \frac{\partial f(x_1,x_2)}{\partial x_2} = f_2(x_1,x_2) = x_1^2$$

The **chain rule** gives the derivative of a function of a function. Thus, if $f(x) \equiv g(h(x))$, then

$$f'(x) = g'(h(x)) \cdot h'(x).$$

This also works for partial derivatives. For example, if $f(x_1,x_2) \equiv g(h(x_1,x_2))$, then

$$\frac{\partial f(x_1,x_2)}{\partial x_1} = g'(h(x_1,x_2)) \cdot \frac{\partial h(x_1,x_2)}{\partial x_1}$$

Similarly, if $f(x_1,x_2) \equiv g(h(x_1,x_2),k(x_1,x_2))$ then

$$\frac{\partial f(x_1,x_2)}{\partial x_1} = g_1(h(x_1,x_2),k(x_1,x_2)) \cdot \frac{\partial h(x_1,x_2)}{\partial x_1} + g_2(h(x_1,x_2),k(x_1,x_2)) \cdot \frac{\partial k(x_1,x_2)}{\partial x_1}$$

The **second derivative** of the function $f(x)$ is written

$$f''(x) \quad \text{or} \quad \frac{d^2 f(x)}{dx^2}$$

and it is obtained by differentiating $f(x)$ twice with respect to x (if you want to determine its value at a particular point $x = x^*$, don't substitute in x^* until *after* you've differentiated twice).

[1] If the material in this section is not *already* familiar to you, you probably won't be able to pass this course.

A **second partial derivative** of a function of two or more variables is analogous, i.e., we will use the expressions

$$f_{11}(x_1,x_2) \quad \text{or} \quad \frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}$$

to denote differentiating twice with respect to x_1 (the notation is analogous when differentiating twice with respect to x_2).

We get a **cross partial derivative** when we differentiate first with respect to x_1 and then with respect to x_2. We will denote this with the expressions

$$f_{12}(x_1,x_2) \quad \text{or} \quad \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}$$

Here's a strange and wonderful result: if we had differentiated in the *opposite order*, that is, first with respect to x_2 and then with respect to x_1, we would have gotten the same result. In other words, we have $f_{12}(x_1,x_2) \equiv f_{21}(x_1,x_2)$ or equivalently $\partial^2 f(x_1,x_2)/\partial x_1 \partial x_2 \equiv \partial^2 f(x_1,x_2)/\partial x_2 \partial x_1$.

B. ELASTICITY

Let the variable y depend upon the variable x according to some function, i.e.:

$$y = f(x)$$

How responsive is y to changes in x? One measure of responsiveness would be to plot the function $f(\cdot)$ and look at its **slope**. If we did this, our measure of responsiveness would be:

$$\text{slope of } f(x) = \frac{\text{absolute change in } y}{\text{absolute change in } x} = \frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} = f'(x)$$

Elasticity is a *different* measure of responsiveness than slope. Rather than looking at the ratio of the *absolute* change in y to the *absolute* change in x, elasticity is a measure of the *proportionate* (or percentage) change in y to the *proportionate* (or percentage) change in x. Formally, if $y = f(x)$, then the **elasticity of y with respect to** x, written $E_{y,x}$, is given by:

$$E_{y,x} = \frac{\text{proportionate change in } y}{\text{proportionate change in } x} = \frac{\left(\frac{\Delta y}{y}\right)}{\left(\frac{\Delta x}{x}\right)} = \left(\frac{\Delta y}{\Delta x}\right)\cdot\left(\frac{x}{y}\right)$$

If we consider very small changes in x (and hence in y), $\Delta y/\Delta x$ becomes $dy/dx = f'(x)$, so we get that the elasticity of y with respect to x is given by:

$$E_{y,x} = \frac{\left(\frac{\Delta y}{y}\right)}{\left(\frac{\Delta x}{x}\right)} = \left(\frac{\Delta y}{\Delta x}\right)\cdot\left(\frac{x}{y}\right) \approx \left(\frac{dy}{dx}\right)\cdot\left(\frac{x}{y}\right) = f'(x)\cdot\left(\frac{x}{y}\right)$$

Note that if $f(x)$ is an increasing function the elasticity will be positive, and if $f(x)$ is a decreasing function, it will be negative.

Recall that since the *percentage change* in a variable is simply 100 times its proportional change, elasticity is also as the ratio of the *percentage change in y* to the *percentage change in x*:

$$E_{y,x} \;=\; \frac{\left(\frac{\Delta y}{y}\right)}{\left(\frac{\Delta x}{x}\right)} \;=\; \frac{100\cdot\left(\frac{\Delta y}{y}\right)}{100\cdot\left(\frac{\Delta x}{x}\right)} \;=\; \frac{\%\text{ change in } y}{\%\text{ change in } x}$$

Since this implies

$$(\%\text{ change in } y) \;=\; E_{y,x}\cdot(\%\text{ change in } x)$$

$E_{y,x}$ is the "conversion factor" between the percentage change in x and the percentage change in y.

Although elasticity and slope are both measures of how responsive y is to changes in x, they are *different* measures. In other words, **elasticity is not the same as slope**. For example, if y is exactly proportional to x, i.e., if we have $y = c\cdot x$, for come constant c, the *slope* of this curve is c, but its *elasticity* is:

$$E_{y,x} \;=\; \left(\frac{dy}{dx}\right)\cdot\left(\frac{x}{y}\right) \;=\; c\cdot\left(\frac{x}{c\cdot x}\right) \;\equiv\; 1$$

In other words, whenever y is exactly proportional to x, the elasticity of y with respect to x will be one, *regardless* of the value of the coefficient c.

Here's another example to show that elasticity is not the same as slope. The function $y = 3 + 4x$ obviously has a constant slope (namely 4). But it does *not* have a constant elasticity:

$$E_{y,x} \;=\; \left(\frac{dy}{dx}\right)\cdot\left(\frac{x}{y}\right) \;=\; 4\cdot\left(\frac{x}{3+4x}\right) \;=\; \left(\frac{4x}{3+4x}\right)$$

which is obviously not constant as x changes.

Finally, we note that if a function has a **constant elasticity**, it must take the form $f(x) \equiv c\cdot x^{\beta}$ for some constants $c > 0$ and β. We prove this by the calculation:

$$E_{f(x),x} \;=\; \frac{d\,f(x)}{dx}\cdot\frac{x}{f(x)} \;=\; \frac{b\cdot c\cdot x^{(b-1)}\cdot x}{c\cdot x^{b}} \;\equiv\; b$$

which yields the constant β for every value of x. (Note that β will be positive if the function is increasing, and negative if it is decreasing.)

C. CONTINUOUS COMPOUNDING AND EXPONENTIAL GROWTH

If a discrete variable X_t grows at rate r and is compounded once per period, its formula is

$$X_t = X_0 \cdot (1+r)^t$$

If X_t still grows at rate r per period but is compounded n times each period, its formula is

$$X_t = X_0 \cdot \left(1+\frac{r}{n}\right)^{nt}$$

What if $X(t)$ is compounded *continuously* (i.e., if $n \to \infty$)? That is, what is the formula for

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$$

To determine this, recall that " log of a limit = limit of the log " so that we have

$$\ln\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}\right) = \lim_{n\to\infty}\left[\ln\left(\left(1+\frac{r}{n}\right)^{nt}\right)\right] = \lim_{n\to\infty}\left[n\cdot t\cdot\ln\left(1+\frac{r}{n}\right)\right]$$

To evaluate this right-hand term, write it as

$$\lim_{n\to\infty}\left[\frac{t\cdot\ln\left(1+r\cdot\frac{1}{n}\right)}{\frac{1}{n}}\right] \quad \text{or equivalently,} \quad \lim_{\omega\to 0}\left[\frac{t\cdot\ln\left(1+r\cdot\omega\right)}{\omega}\right] \quad \text{where} \quad \omega=\tfrac{1}{n}$$

Since both numerator and denominator go to zero as $\omega \to 0$, we must use l'Hopital's rule:

$$\lim_{\omega\to 0}\left[\frac{t\cdot\ln\left(1+r\cdot\omega\right)}{\omega}\right] = \frac{\lim_{\omega\to 0}\left[t\cdot\frac{\partial\ln(1+r\cdot\omega)}{\partial\omega}\right]}{\lim_{\omega\to 0}\left[\frac{\partial\omega}{\partial\omega}\right]} = \frac{\lim_{\omega\to 0}\left[t\cdot\frac{r}{\left(1+r\cdot\omega\right)}\right]}{1} = r\cdot t$$

Thus, we've proven

$$\ln\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}\right) = r\cdot t$$

Taking the exponential of both sides of this equation gives us

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$

So the formula for a variable that compounds *continuously* at rate r per period is:

$$X(t) \equiv X_0\cdot e^{r\cdot t}$$

A variable with this formula is said to have a **constant proportional growth rate of *r***, or to exhibit **exponential growth**. Observe that it satisfies the constant proportional growth formula $X'(t) \underset{t}{\equiv} r\cdot X(t)$.

D. PROPORTIONAL GROWTH RATES & CONSTANT PROPORTIONAL GROWTH

If $X(t)$ is a continuous-time variable, we must distinguish between its *absolute rate of increase*, that is to say, $X'(t) = dX(t)/dt$, and its *proportional growth rate*, which is given by the formula

$$\frac{X'(t)}{X(t)} = \frac{dX(t)/dt}{X(t)}$$

To see the difference between the absolute rate of increase of a variable and it's proportional growth rate, consider the variable

$$X(t) \equiv c \cdot t \qquad \text{for some constant } c > 0$$

Its absolute rate of increase is clearly $X'(t) = dX(t)/dt = c$, which is constant as time proceeds. But its *proportional rate of growth* is given by

$$\frac{X'(t)}{X(t)} = \frac{dX(t)/dt}{X(t)} = \frac{c}{c \cdot t} = \frac{1}{t}$$

which is clearly *decreasing* as time proceeds.

If $X(t) \equiv c \cdot t$ (or $X(t) \equiv a + c \cdot t$) is the formula for a variable that has a *constant absolute rate of increase,* what is the formula for a variable that has a *constant proportional growth rate?* The answer is simple: $X(t)$ will have a constant proportional growth rate if and only if it exhibits *exponential growth* (as defined in the previous section), or in other words, if and only if it is given by the formula

$$X(t) \equiv X_0 \cdot e^{rt}$$

It is remarkably easy to prove this. We simply have:

$$\frac{X'(t)}{X(t)} = \frac{dX(t)/dt}{X(t)} = \frac{r \cdot X_0 \cdot e^{rt}}{X_0 \cdot e^{rt}} \equiv r$$

Thus "compounding continuously at rate *r*" as defined in the previous section is exactly the same thing as a **constant proportional growth rate of *r***. A very useful thing to know.

E. PRODUCTS AND RATIOS OF GROWING VARIABLES

Consider a continuous-time variable $X(t)$ and a continuous-time variable $Y(t)$, and let $Z(t) \equiv X(t) \cdot Y(t)$ be the *product* of $X(t)$ and $Y(t)$. How does the instantaneous proportional growth rate of $Z(t)$ at some time t relate to the instantaneous proportional growth rates of $X(t)$ and $Y(t)$ at time t?

The answer is that $Z(t)$'s instantaneous proportional growth rate will be given by the *sum* of the instantaneous proportional growth rates of $X(t)$ and $Y(t)$. To show this, we simply calculate the formula for $Z(t)$'s instantaneous growth rate, namely the formula for $Z'(t)/Z(t)$:

$$\frac{Z'(t)}{Z(t)} = \frac{\frac{d[X(t) \cdot Y(t)]}{dt}}{X(t) \cdot Y(t)} = \frac{X'(t) \cdot Y(t) + X(t) \cdot Y'(t)}{X(t) \cdot Y(t)} = \frac{X'(t)}{X(t)} + \frac{Y'(t)}{Y(t)}$$

A special case of this result is the case when $X(t)$ and $Y(t)$ happen to have *constant* proportional growth rates of r and s respectively, so that they are given by the formulas

$$X(t) \equiv X_0 \cdot e^{r \cdot t} \quad \text{and} \quad Y(t) \equiv Y_0 \cdot e^{s \cdot t}$$

In which case we find $\quad Z(t) \equiv X(t)\cdot Y(t) \equiv (X_0 \cdot Y_0)\cdot e^{(r+s)\cdot t}$

What about when $Z(t) \equiv X(t)/Y(t)$ is the *ratio* of the variables $X(t)$ and $Y(t)$? In this case, $Z(t)$'s instantaneous proportional growth rate will be given by the *difference* of the instantaneous proportional growth rates of $X(t)$ and $Y(t)$. To show this, Once again, we simply calculate the formula for $Z(t)$'s instantaneous growth rate:

$$\frac{Z'(t)}{Z(t)} = \frac{\frac{d[X(t)/Y(t)]}{dt}}{X(t)/Y(t)} = \frac{[X'(t)\cdot Y(t) - X(t)\cdot Y'(t)]/Y(t)^2}{X(t)\cdot Y(t)}$$

$$= \frac{[X'(t)\cdot Y(t)]/Y(t)^2}{X(t)\cdot Y(t)} - \frac{[X(t)\cdot Y'(t)]/Y(t)^2}{X(t)\cdot Y(t)} = \frac{X'(t)}{X(t))} - \frac{Y'(t)}{Y(t)} = r - s$$

A special case of this result is when $X(t)$ and $Y(t)$ have constant proportional growth rates, so they are given by the formulas

$$X(t) \equiv X_0 \cdot e^{rt} \quad \text{and} \quad Y(t) \equiv Y_0 \cdot e^{st}$$

so that

$$\frac{X(t)}{Y(t)} \equiv \frac{X_0 \cdot e^{rt}}{Y_0 \cdot e^{st}} \equiv \left(\frac{X_0}{Y_0}\right)\cdot e^{(r-s)\cdot t}$$

F. FUNCTIONS OF GROWING VARIABLES

Finally, instead of just the simple products or ratios, we can also calculate the growth behavior of *general functions* of a growing variable $X(t)$. Consider the general function $F(X)$. Since

$$\frac{dF(X(t))}{dt} = \frac{dF(X)}{dX}\cdot\frac{dX(t)}{dt}$$

we have

$$\frac{\left(\frac{dF(X(t))}{dt}\right)}{F(X(t))} = \left(\frac{dF(X)}{dX}\cdot\frac{X(t)}{F(X(t))}\right)\cdot\left(\frac{dX(t)/dt}{X(t)}\right) = E_{F(X),X}\cdot\left(\frac{dX(t)/dt}{X(t)}\right)$$

i.e., the growth rate of the variable $F(X(t))$ is just the *elasticity of* $F(X)$ *with respect to* X multiplied by the growth rate of $X(t)$. This is intuitive: Say that in each period, $X(t)$ grows by (approximately) r%. Since each percentage point increase in $X(t)$ yields a $E_{F(X),X}$ percentage point increase in $F(X)$, the growth of $F(X(t))$ in a period will be the product of these two terms.

Now consider the general function $F(X,Y)$ of *two* growing variables $X(t)$ and $Y(t)$. Since

$$\frac{dF(X(t),Y(t))}{dt} = \frac{dF(X,Y)}{dX}\cdot\frac{dX(t)}{dt} + \frac{dF(X,Y)}{dY}\cdot\frac{dY(t)}{dt}$$

we have

$$\frac{\dfrac{dF(X(t),Y(t))}{dt}}{F(X(t),Y(t))} = \frac{\dfrac{dF(X,Y)}{dX}\cdot\dfrac{dX(t)}{dt}}{F(X(t),Y(t))} + \frac{\dfrac{dF(X,Y)}{dY}\cdot\dfrac{dY(t)}{dt}}{F(X(t),Y(t))}$$

$$= \left(\frac{dF(X,Y)}{dX}\cdot\frac{X(t)}{F(X(t),Y(t))}\right)\cdot\left(\frac{dX(t)/dt}{X(t)}\right) + \left(\frac{dF(X,Y)}{dY}\cdot\frac{Y(t)}{F(X(t),Y(t))}\right)\cdot\left(\frac{dY(t)/dt}{Y(t)}\right)$$

$$= E_{F(X,Y),X}\cdot\left(\frac{dX(t)/dt}{X(t)}\right) + E_{F(X,Y),Y}\cdot\left(\frac{dY(t)/dt}{Y(t)}\right)$$

So that the growth rate of a function of variables is a *weighted sum* of the growth rates of the arguments, where the weights are the elasticities of the function with respect to the arguments. But recall that when the function $F(X,Y)$ exhibits *constant returns to scale*, we have

$$E_{F(X,Y),X} + E_{F(X,Y),Y} \equiv 1$$

I.e., the elasticities of $F(X,Y)$ with respect to the X and Y add to unity, so that the growth rate of the function $F(X,Y)$ will be a *weighted average* of the growth rates of X and Y, with the weights given by the elasticities $E_{F(X,Y),X}$ and $E_{F(X,Y),Y}$

G. CONSTANT RETURNS TO SCALE PRODUCTION FUNCTIONS

A production function $F(L,K)$ exhibits **constant returns to scale** (CRS) if it satisfies the identity[2]

(*) $$F(\lambda L,\lambda K) \equiv \lambda\cdot F(L,K) \qquad \text{for all } L, K, \lambda \geq 0$$

In other words, scaling both labor and capital up (or down) by some factor λ leads to output also being scaled up (or down) by the factor λ. Thus, an exact *doubling* of both labor and capital will lead to an exact *doubling* of output, a *tripling* of both labor and capital will lead to a *tripling* of output, a *halving* of both labor and capital will lead to a *halving* of output, etc.

Constant returns to scale production functions exhibit three strange and wonderful properties:

1. Marginal and Average Products are Scale Invariant: If $F(L,K)$ is a constant returns to scale production function, then all of its four related functions, namely:

average product of labor:	$AP_L(L,K) = F(L,K)/L$
average product of capital:	$AP_K(L,K) = F(L,K)/K$
marginal product of labor:	$MP_L(L,K) = \partial F(L,K)/\partial L$
marginal product of capital:	$MP_K(L,K) = \partial F(L,K)/\partial K$

are **scale invariant**. This means that scaling both labor and capital up (or down) by a common factor λ leads to *no change* in AP_L, AP_K, MP_L or MP_K.

[2] Throughout this section, the identity sign "≡" will indicate an identity with respect to the variables L, K and λ.

To prove this for the average products, simply divide the identity (*) by λL and by λK, to get:

$$F(\lambda L,\lambda K)/\lambda L \;\equiv\; F(L,K)/L \qquad \text{and} \qquad F(\lambda L,\lambda K)/\lambda K \;\equiv\; F(L,K)/K$$

in other words:

$$AP_L(\lambda L,\lambda K) \;\equiv\; AP_L(L,K) \qquad \text{and} \qquad AP_K(\lambda L,\lambda K) \;\equiv\; AP_K(L,K)$$

but this says precisely that a doubling (or tripling, or halving...) of both L and K has no effect on either the average product of labor or the average product of capital.

To prove this for marginal products, differentiate the identity (*) with respect to L and to K:

$$\frac{\partial F(\lambda L,\lambda K)}{\partial(\lambda L)}\cdot\lambda \;\equiv\; \lambda\cdot\frac{\partial F(L,K)}{\partial L} \qquad \text{and} \qquad \frac{\partial F(\lambda L,\lambda K)}{\partial(\lambda K)}\cdot\lambda \;\equiv\; \lambda\cdot\frac{\partial F(L,K)}{\partial K}$$

in other words:

$$MP_L(\lambda L,\lambda K) \;\equiv\; MP_L(L,K) \qquad \text{and} \qquad MP_K(\lambda L,\lambda K) \;\equiv\; MP_K(L,K)$$

but this says precisely that a doubling (or tripling, or halving...) of both L and K has no effect on either the marginal product of labor or the marginal product of capital.

2. <u>Euler's Theorem</u>: Euler's theorem states that if $F(L,K)$ is constant returns to scale, then

$$F(L,K) \;\equiv\; F_L(L,K)\cdot L \;+\; F_K(L,K)\cdot K \qquad \text{for all } L,\, K,\, \lambda \geq 0$$

or equivalently

$$F(L,K) \;\equiv\; MP_L(L,K)\cdot L \;+\; MP_K(L,K)\cdot K \qquad \text{for all } L,\, K,\, \lambda \geq 0$$

Euler's Theorem links the formula for the production function $F(L,K)$ to the formulas for its marginal products $MP_L(L,K)$ and $MP_K(L,K)$. Although it will be extremely useful, it doesn't really have a good intuitive interpretation. Accordingly, we present three different proofs:

Proof #1: Constant returns to scale implies

$$\lambda\cdot F(L,K) \;\equiv\; F(\lambda L,\lambda K)$$

Differentiate this identity with respect to λ to get the identity

$$F(L,K) \;\equiv\; F_L(\lambda L,\lambda K)\cdot L \;+\; F_K(\lambda L,\lambda K)\cdot K$$

evaluating this identity at $\lambda = 1$ yields

$$F(L,K) \;\equiv\; F_L(L,K)\cdot L \;+\; F_K(L,K)\cdot K$$

<u>Q.E.D.</u>

Proof #2: First, *calculus* implies that if L and K change by small amounts ΔL and ΔK, we have

$$\Delta Q \;=\; MP_L(L,K)\cdot\Delta L \;+\; MP_K(L,K)\cdot\Delta K$$

Second, *constant returns to scale* implies that if L and K each increase by a common proportion γ, then Q will also increase by proportion γ. In other words, if $\Delta L = \gamma\cdot L$ and $\Delta K = \gamma\cdot K$, then $\Delta Q = \gamma\cdot Q$. Substituting these into the above equation yields

$$\gamma\cdot Q \;=\; MP_L(L,K)\cdot(\gamma\cdot L) \;+\; MP_K(L,K)\cdot(\gamma\cdot K)$$

hence

$$Q \;=\; MP_L(L,K)\cdot L \;+\; MP_K(L,K)\cdot K$$

<u>Q.E.D.</u>

Proof #3: Since $F(\lambda L,\lambda K) \;\equiv\; \lambda\cdot F(L,K)$ for all L, K, $\lambda \geq 0$

we also have $((1+\lambda)\cdot L,(1+\lambda)\cdot K) \;\equiv\; (1+\lambda)\cdot F(L,K)$

Subtract $F(L,K)$ from both sides to get:

$$F((1+\lambda)\cdot L,(1+\lambda)\cdot K) - F(L,K) \;\equiv\; \lambda\cdot F(L,K)$$

Add and subtract $F(L,(1+\lambda)\cdot K)$ from the left side to get:

$$F((1+\lambda)\cdot L,(1+\lambda)\cdot K) - F(L,(1+\lambda)\cdot K) + F(L,(1+\lambda)\cdot K) - F(L,K) \;\equiv\; \lambda\cdot F(L,K)$$

Divide everything by λ to get

$$\frac{F\big((1+\lambda)\cdot L,(1+\lambda)\cdot K\big) \;-\; F\big(L,(1+\lambda)\cdot K\big)}{\lambda} \;+\; \frac{F\big(L,(1+\lambda)\cdot K\big) \;-\; F(L,K)}{\lambda} \;\;\equiv\;\; F(L,K)$$

or equivalently:

$$\frac{F(L+\lambda\cdot L,K+\lambda\cdot K) - F(L,K+\lambda\cdot K)}{\lambda\cdot L}\cdot L \;+\; \frac{F(L,K+\lambda\cdot K) - F(L,K)}{\lambda\cdot K}\cdot K \;\equiv\; F(L,K)$$

As $\lambda \to 0$, this becomes $F_L(L,K)\cdot L \;+\; F_K(L,K)\cdot K \;=\; F(L,K)$ **<u>Q.E.D.</u>**

3. <u>Input Elasticities Sum to Unity</u>: The final strange and wonderful property of constant returns to scale production functions concern the following two elasticities:

the elasticity of output with respect to labor: $E_{F(L,K),L}$

the elasticity of output with respect to capital: $E_{F(L,K),K}$

To derive this property, recall that Euler's theorem implies

$$F_L(L,K)\cdot L \;+\; F_K(L,K)\cdot K \;\equiv\; F(L,K) \qquad \text{for all } L,\, K,\, \lambda \geq 0$$

Dividing through by $F(L,K)$ yields

$$F_L(L,K)\cdot\frac{L}{F(L,K)} \;\;+\;\; F_K(L,K)\cdot\frac{K}{F(L,K)} \;\;\equiv\;\; 1$$

i.e., $E_{F(L,K),L}(L,K) \;+\; E_{F(L,K),K}(L,K) \;\equiv\; 1$

In other words, if the production function exhibits constant returns to scale, the elasticity of output with respect to labor and the elasticity of output with respect to capital always add to one. The intuition behind this is that a 1% rise in L affects output by the percentage amount $E_{F,L}$, and a 1% rise in K affects output by the percentage amount $E_{F,K}$, so a 1% rise in *both* L and K should affect output by the sum of these two amounts. But constant returns to scale implies that a 1% rise in both L and K should affect output by exactly 1%, so the sum of these two amounts must be exactly one.